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A unified kinetic theory approach to external rarefied gas flows. Part 1. Derivation of hydrodynamic equations

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A set of partial differential equations of the Navier–Stokes type is derived for external rarefied gas flows at all Knudsen numbers. Only the expressions of the stress tensor and heat flux vector are different from the customary Navier–Stokes relations and Fourier law. The new expressions are calculated from an approximate distribution function constructed through analysis of the BGK model of the Boltzmann equation so that it retains the qualitative property of the collision term and reproduces accurately the two extreme continuum and free-molecule regimes. Explicit forms of the stress tensor and heat flux components are given for low-speed two-dimensional flows. Solutions for vanishing Mach number and compressibility effects are then discussed. For a nearly isothermal cylinder, the present formulation leads to only one governing equation of the Navier–Stokes type for all flow regimes.

1. Introduction

Rarefied gas flows past solid bodies have recently been investigated by many authors because of their theoretical interest and obvious application to earth satellite experiments and upper atmosphere measurements. Starting from the Boltzmann equation, different approaches were devised to account for its rarefaction effects. Logically, rational simplifications are possible only in the limiting cases. For example, in the near-continuum regime, Tamada & Yamamoto (1967) used the linearized BGK equation to describe the flow past a circular cylinder at small Reynolds number and obtained first- and second-order terms. In the other extreme of the nearly free-molecule regime Liu, Pang & Jew (1965) and later Willis (1965) employed the technique of Knudsen iteration to calculate the drag of a sphere.

In the transition regime, approximate schemes are invariably of an *ad hoc* nature. We cite here the work of Liu & Passamaneck (1967) and Liu & Sugimura (1969), who applied Lees' moment method (Lees 1959) to rarefied flow past a cylinder and a sphere. Lees' approach consists of dividing the phase space according to a line-of-sight principle and taking a different Maxwellian-type velocity distribution function in each region to calculate the moments. Admirably simple this method has achieved wide popularity. It seems suitable as an approximation for nearly free-molecule flows dominated by the geometric shadow effect. However, for moderate and small Knudsen numbers, there are regions in the flow where velocity and temperature gradient effects are relatively important and the distribution function might be poorly approximated by means of Maxwellian components. Above all, it does not give the correct continuum limit. Thus, in the case of a flow past a circular cylinder Liu & Passamaneck's derivation of the drag formula seems to be semi-empirical, and the velocity field, as well as the drag, does not exhibit the usual logarithmic behaviour one witnesses in the continuum regime to any order. These details are lost because of the oversimplification.

A satisfactory unified kinetic theory approach to the problem of an unbounded rarefied flow past a body, at all Knudsen numbers Kn, should contain a comprehensive and smooth description of the transition of the velocity distribution function that is discontinuous near the boundary but tends to the Chapman–Enskog solution in the continuum limit at large distances. With this feature in mind, Shen (1963, 1966) proposed a different approach, emphasizing in addition an extension of the customary hydrodynamic equations through modification of the constitutive relations (Navier–Stokes and Fourier law) as well as the boundary conditions. The rarefied gas is thus treated as a non-Newtonian fluid. In Shen's theory the new constitutive relations and the boundary conditions are obtained from an approximate distribution function suggested by an analysis of the BGK model of the Boltzmann equation. The approximate form is constructed so that: (i) it reproduces accurately and smoothly the two extreme free-molecule and continuum regimes; (ii) the order of the new set of hydrodynamic equations is the same as that of the Navier–Stokes equations; (iii) it accounts for the geometry of the problem dealt with. Shen also gave a simple form of such a distribution function and the results of an application, due to Lo (1964), to the heat transfer between two parallel plates with a large temperature difference. Preliminary results using Shen's approach to the problem of a flow past a circular cylinder were given by Atassi & Shen (1969). Later, Cercignani & Trioni (1969) proposed an extension of the Navier–Stokes equations with the sole modification on the boundary conditions. They borrowed the classical Maxwellian idea (Maxwell 1965) of evaluating the distribution function prevailing at one mean free path far from the wall to describe the inner conditions. This is tantamount to 'patching' the inner region (free-molecule like) solution directly to that of the far-field continuum regime without taking any account of the behaviour in the transition region. As mentioned by these authors, because of the geometric shadow effect this approach is less accurate for two-dimensional than for three-dimensional flows. Besides, the proposed boundary conditions are very complicated and the approach appears to be less consistent than the one put forward by Shen.

In part 1 of this paper, we develop Shen's approach to derive a set of generalized hydrodynamic equations for external rarefied gas flows. Later, we linearize the rarefied terms of these equations with respect to velocity and give a full discussion of compressibility effects for all rarefied gas regimes, based on the explicit form of the stress tensor and heat flux vector for two-dimensional flows. In part 2, we treat the problem of a low-speed rarefied gas flow past a circular cylinder, giving a fuller account than that for our preliminary results. Asymptotic solutions for $Kn \ll 1$ and $Kn \gg 1$ are deduced using the method of matched asymptotic expansions, and are compared with other theories and known experimental data on the drag and heat-transfer coefficients.

2. Construction of the approximate distribution function

To establish the essential features of a suitable approximate distribution function we examine the BGK model of the Boltzmann equation

$$dF/dt = -K(F-F^{(0)}), \tag{1}$$

where $d/dt \equiv (\partial/\partial t + \boldsymbol{\xi}.\nabla)$, F is the velocity distribution function, $F^{(0)}$ the corresponding Maxwellian, K the collision frequency, and $\boldsymbol{\xi}$ stands for the molecular velocity vector. An integral form of (1) is given by

$$F(t,\mathbf{x},\boldsymbol{\xi}) = F(t_0,\mathbf{x}-\boldsymbol{\xi}(t-t_0),\boldsymbol{\xi})\,E(t,t_0) + \int_{t_0}^{t} K(\tau,\mathbf{x}-\boldsymbol{\xi}(t-\tau))\,F^{(0)}(\tau,\mathbf{x}-\boldsymbol{\xi}(t-\tau),\boldsymbol{\xi})\,E(t,\tau)\,d\tau, \tag{2}$$

where $\mathbf{x}$ is the space co-ordinate vector, t_0 and t are the initial and final times respectively and

$$E(t,t_1) = \exp\left[-\int_{t_1}^{t} K(\tau,\mathbf{x}-\boldsymbol{\xi}(t-\tau))\,d\tau\right]. \tag{3}$$

We define the following non-dimensional quantities:

$$K^* = \frac{\lambda}{(2RT)^{\frac{1}{2}}}K,\quad \xi^* = \frac{\xi}{(2RT)^{\frac{1}{2}}},\quad S^0 = \frac{S}{\lambda},$$

where λ, T and ξ are the mean free path, the temperature of the gas and the magnitude of the molecular velocity respectively. S stands for the distance travelled by the molecules in phase trajectory and R is the gas constant.

To examine the effect of the attenuation factor $E(t,t_1)$ on the behaviour of the distribution function F as the molecules move away from the initial boundary we restrict ourselves to the steady state. Noting that $dt = dS/\xi$, (2) and (3) become

$$F(S^0,\boldsymbol{\xi}) = F(S^0_w,\boldsymbol{\xi})\,E(S^0,S^0_w) + \frac{\overline{K}^*}{\xi^*}\int_{S^0_w}^{S^0} F^{(0)}(S_1,\boldsymbol{\xi})\,E(S^0,S^0_1)\,dS^0_1, \tag{4}$$

$$E(S^0,S^0_1) = \exp\left[-(\overline{K}^*/\xi^*)\,(S^0-S^0_1)\right], \tag{5}$$

where the subscript w denotes values at the wall and $\overline{K}^*$ is some average value of K^*. Equations (4) and (5) show clearly that whatever the degree of rarefaction of the gas may be the distance $S^0 - S^0_w$ travelled by the molecules from the boundary in terms of mean free paths plays the most important role in determining the form of the distribution function $F(S^0,\boldsymbol{\xi})$. Thus, we make the following observations.

(i) For $S^0 - S^0_w \ll 1$

$$F(S^0,\boldsymbol{\xi}) = F(S^0_w,\boldsymbol{\xi}) - (\overline{K}^*/\xi^*)\,[F(S^0_w,\boldsymbol{\xi}) - F^{(0)}(S^0_w,\boldsymbol{\xi})]\,(S^0-S^0_w) + \dots .$$

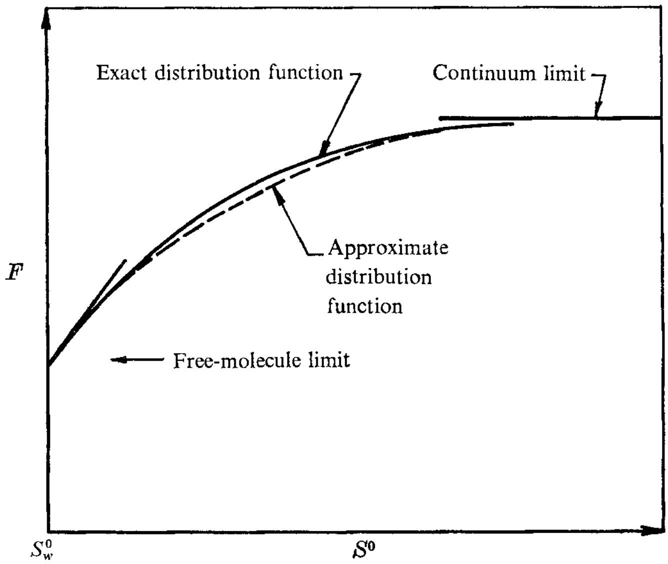

FIGURE 1. A schematic sketch of the construction of the approximate velocity distribution function. ———, exact velocity distribution function as given by (9) versus the distance S^0 travelled by the molecules in phase trajectory from the initial boundary S^0_w; ---, approximate form of this velocity distribution function defined by (10), when the correction $\tilde{F}$ is omitted.

The velocity distribution function keeps essentially the form assigned to it at the boundary, with a slight modification proportional to $(S^0 - S^0_w)$.

(ii) For $S^0 - S^0_w \gg 1$, $E(S^0, S^0_w)$ dies out exponentially and so does $F(S^0_w, \xi)$. When the space point under consideration satisfies this requirement for all boundaries we expect the resulting distribution function to correspond to the Chapman–Enskog solution suitable for the continuum regime.

(iii) $S^0 - S^0_w = O(1)$, both terms in (4) are equally important.

To cover this intermediary region Shen (1963) cast (4) in the following form:

$$F = F_w E + F_c(1 - E) + \tilde{F}, \tag{6}$$

where F_c is the Chapman–Enskog distribution function and $\tilde{F}$ is a correction which vanishes at both the limits $S^0 - S^0_w \to 0$ and $S^0 - S^0_w \to \infty$, and which Shen neglected, for simplicity in the calculation of the shear stress tensor and the heat flux vector.

In the following, we use a different form of (6) by further imposing the condition that not only $\tilde{F} \to 0$, but also $d\tilde{F}/dS^0 \to 0$ as $S^0 \to S^0_w$. We also evaluate the correction $\tilde{F}$. A similar modification has already been adopted by Shen (1967) in the context of radiative heat transfer. The additional complexity is minor, but the improvement of accuracy appears significant. To visualize our approximation let us imagine we were able to solve the Boltzmann equation and obtain the exact velocity distribution function for a given problem. Figure 1 shows schematically this exact velocity distribution function as a solid line and the approximate distribution function we propose to construct as a dashed line, plotted against the distance S^0 travelled by the molecules in phase trajectory from the initial boundary S^0_w.

We start with (2) and try to obtain the Chapman–Enskog distribution function F_c for $t-t_0 \gg 1/K$. Within the framework of (1), F_c is obtained by evaluating the left-hand side using $F^{(0)}$:

$$F_c = F^{(0)} - \frac{1}{K}\frac{d}{dt}F^{(0)}. \tag{7}$$

Since F_c is defined by the local gradient of $F^{(0)}$ we may expand $F^{(0)}$ in the neighbourhood of t, along the trajectory of the molecules:

$$F^{(0)}(\tau) = F^{(0)}(t) + (\tau - t)\frac{d}{dt}F^{(0)}(t) + \frac{(\tau-t)^2}{2!}\frac{d^2}{dt^2}F^{(0)}(\tau_1), \tag{8}$$

where $\tau < \tau_1 < t$. It is worth noting that though the function $F^{(0)}$ is regular its derivatives at the wall may become very large. However, as shown by Welander (1954), this singularity is weak and the integrated values are moderate. Using (8) to evaluate the integral of (2) and assuming for simplicity a single average collision frequency $\bar{K}$, we get

$$\begin{aligned} F^{(0)}(t,\mathbf{x},\boldsymbol{\xi}) = F(t_0, \mathbf{x}-\boldsymbol{\xi}(t-t_0), \boldsymbol{\xi})\, E(t,t_0) + F_c(t,\mathbf{x},\boldsymbol{\xi})\,[1-E(t,t_0)] \\ + (t-t_0)\, d/dt\, F^{(0)}(t,\mathbf{x},\boldsymbol{\xi})\, E(t,t_0) + \tilde{F}, \end{aligned} \tag{9}$$

where $\tilde{F} = O[(t-t_0)^2\, d^2/dt^2\, F^{(0)}(t)\, E(t,t_0)]$. It is easy to verify that $\tilde{F}$ and $d\tilde{F}/dt$, vanish as $t-t_0 \to 0$. To avoid Burnett-like terms of the continuum regime $\tilde{F}$ will be omitted for all flow regimes. It is more convenient to rewrite (9) as

$$F(t,\mathbf{x},\boldsymbol{\xi}) = F_c(t,\mathbf{x},\boldsymbol{\xi}) + F_R(t,t_0;\mathbf{x},\boldsymbol{\xi})\, E(t,t_0), \tag{10}$$

where the 'rarefied component' F_R is then

$$F_R = F(t_0, \mathbf{x}-\boldsymbol{\xi}(t-t_0), \boldsymbol{\xi}) - \left[F_c(t,\mathbf{x},\boldsymbol{\xi}) - (t-t_0)\frac{d}{dt}F^{(0)}(t,\mathbf{x},\boldsymbol{\xi})\right]. \tag{11}$$

At each point of the flow $M(x,y,z)$, the velocity distribution function is, in general, discontinuous along the conical surface subtended by the body (figure 2). This surface of discontinuity divides the velocity space into two semi-spaces A and B. Then, the distribution function of molecules whose velocity $\boldsymbol{\xi} \in A$ is

$$F(t,\mathbf{x},\boldsymbol{\xi}) = F_c(t,\mathbf{x},\boldsymbol{\xi}) + F_R(t,t_w;\mathbf{x},\boldsymbol{\xi})\, E(t,t_w), \tag{12}$$

where t_w is the time at which these molecules left the surface of the body. Whereas, for those molecules whose velocity is $\boldsymbol{\xi} \in B$, the corresponding $(t-t_0)$ is very large, and we simply have

$$F(t,\mathbf{x},\boldsymbol{\xi}) = F_c(t,\mathbf{x},\boldsymbol{\xi}). \tag{13}$$

It may be emphasized that our answer is not meant to be an approximation of the BGK model, in spite of the presence of the collision frequency K. Note, for example, the insertion of F_c to replace its counterpart in the BGK version.

3. Derivation of hydrodynamic equations for external rarefied flows

We define the average value $\langle Q\rangle$ of any molecular property $Q(\boldsymbol{\xi})$ by

$$\langle Q\rangle = \frac{1}{n(t,\mathbf{x})}\int_{\boldsymbol{\xi}} Q(\boldsymbol{\xi})\, F(t,\mathbf{x},\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{14}$$

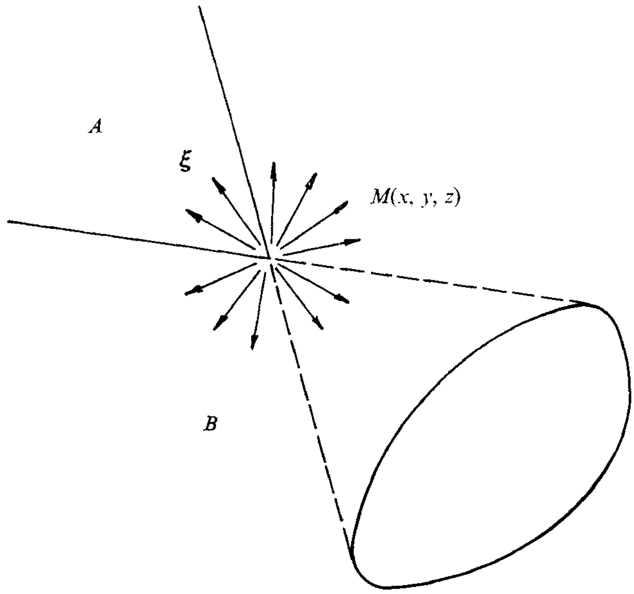

FIGURE 2. The conical surface of discontinuity in the phase space.

where the integration is carried over all the phase space and $n(t, \mathbf{x})$ is the molecular density of the gas defined by (14) when $Q(\boldsymbol{\xi}) \equiv 1$. Substituting the expression for $F(t, \mathbf{x}, \boldsymbol{\xi})$ given by (12) and (13) into (14), we get, after rearrangement,

$$n(t, \mathbf{x}) \langle Q \rangle = \int_{\boldsymbol{\xi}\,\text{all phase space}} F_c(t, \mathbf{x}, \boldsymbol{\xi})\, Q(\boldsymbol{\xi})\, d\boldsymbol{\xi} + \int_{\boldsymbol{\xi}\,\text{semi-phase space}\,A} F_R(t, \mathbf{x}, \boldsymbol{\xi})\, Q(\boldsymbol{\xi})\, E(t, t_w)\, d\boldsymbol{\xi}. \quad (15)$$

The first integral of (15) will give the Navier–Stokes value of $n(t, \mathbf{x}) \langle Q \rangle$, which we denote by $n^{(0)}(t, \mathbf{x})\, Q_C$, where $n^{(0)}(t, \mathbf{x})$ is a normalizing factor and has the significance of a partial density. The last integral in (15) defines a complementary value for $n(t, \mathbf{x}) \langle Q \rangle$, which we denote by $n^{(0)}(t, \mathbf{x})\, Q_R$. Hence (15) may be written as

$$n(t, \mathbf{x}) \langle Q \rangle = n^{(0)}(t, \mathbf{x})\, [Q_C(t, \mathbf{x}) + Q_R(t, \mathbf{x})]. \quad (16)$$

Equation (16) states that the mean value of any molecular property $Q(\boldsymbol{\xi})$ is the sum of the continuum Navier–Stokes value plus a term due to rarefaction. Note that in Q_R both the geometric 'shadow effect' and the dynamical collision frequency consequences are approximately accounted for by using (15).

It is obvious that the collision integral of the Boltzmann equation is not, in general, conservative for the five summational invariants when the distribution functions defined by (12) and (13) are used. A mixed method is thus used. The equations of change of molecular properties, when these are the five summational invariants $m, m\xi_i$ and $\frac{1}{2}m\xi_i\xi_i$, are derived by the use of the unknown exact solution of the Boltzmann equation (see Chapman & Cowling 1964, pp. 51, 52). Only the components P_{ij} of the stress tensor and $\dot{q}_i$ of the heat flux vector as well as the boundary conditions at the body required for the solution of these equations are calculated from (15) by giving to $Q(\boldsymbol{\xi})$ the appropriate values. Thus, we arrive at the following generalized hydrodynamic equations:

$$\left.\begin{aligned} &\frac{Dn}{Dt}+n\frac{\partial u_i}{\partial x_i}=0,\\ &n\frac{D}{Dt}u_j-\frac{\partial}{\partial x_i}(P_{ij})_C=\frac{\partial}{\partial x_i}(P_{ij})_R,\\ &nC_v\frac{D}{Dt}T-(P_{ij})_C\frac{\partial u_i}{\partial x_j}+\frac{\partial}{\partial x_i}(\dot{q}_i)_C=(P_{ij})_R\frac{\partial u_i}{\partial x_j}-\frac{\partial}{\partial x_i}(\dot{q}_i)_R, \end{aligned}\right\} \tag{17}$$

where $D/Dt \equiv \partial/\partial t + u_i\,\partial/\partial x_i$, and u_i and T are the gas velocity and temperature respectively. $(P_{ij})_C$ is the continuum stress tensor given by the Navier–Stokes relationships and $(\dot{q}_i)_C$ is the continuum heat flux vector given by Fourier's law:

$$\left.\begin{aligned} (P_{ij})_C &= \mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\left(P+\tfrac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right)\delta_{ij},\\ (\dot{q}_i)_C &= -k\,\partial T/\partial x_i, \end{aligned}\right\} \tag{18}$$

where $P = nRT$ is the scalar pressure, δ_{ij} is the Kronecker delta and μ and k are the coefficients of viscosity and thermal conductivity respectively. Finally, $(P_{ij})_R$ and $(\dot{q})_R$ are the complementary parts of the stress tensor and heat flux vector due to rarefaction effects:

$$\left.\begin{aligned} (P_{ij})_R &= -\int_{\boldsymbol{\xi}\in A}(\xi_i-u_i)(\xi_j-u_j)\,F_R(t,\mathbf{x},\boldsymbol{\xi})\,E(t,t_w)\,d\boldsymbol{\xi},\\ (\dot{q}_i)_R &= \frac{1}{2}\int_{\boldsymbol{\xi}\in A}(\xi_i-u_i)(\xi_j-u_j)(\xi_j-u_j)\,F_R(t,\mathbf{x},\boldsymbol{\xi})\,E(t,t_w)\,d\boldsymbol{\xi}. \end{aligned}\right\} \tag{19}$$

We note that the five partial differential equations forming the system (17) contain in addition to the usual five hydrodynamic unknown functions, n, u_i and T, the partial density function $n^{(0)}$ associated with the Chapman–Enskog distribution function F_c, and the velocity at the wall n_w. An additional equation is then needed. This equation is obtained by putting $Q = 1$ in (15):

$$n(t,\mathbf{x}) = n^{(0)}(t,\mathbf{x})+\int_{\boldsymbol{\xi}\in A}E(t,t_w)\,F_R(t,\mathbf{x},\boldsymbol{\xi})\,d\boldsymbol{\xi}. \tag{20}$$

The boundary conditions required to solve (17) and determine the value of n_w are obtained by using (15) to calculate the gas velocity and temperature at the wall:

$$\left.\begin{aligned} (nu_i)_{gw} &= (n^{(0)}u_i)_{gw}+\int_{\boldsymbol{\xi}\in A}\xi_i F_R\,d\boldsymbol{\xi},\\ (3nRT)_{gw} &= (3n^{(0)}RT)_{gw}+\int_{\boldsymbol{\xi}\in A}(\xi_i-u_i)^2F_R\,d\boldsymbol{\xi}, \end{aligned}\right\} \tag{21}$$

where the subscript gw denotes the condition of the gas at the wall. The condition of an impermeable wall and the slip velocity can be immediately deduced from (21) by introducing the geometry of the body. Finally, our equations depend on the collision frequency K. From dimensional considerations, it appears that

$$K = \alpha P/\mu, \tag{22}$$

where α is a numerical parameter. The value assigned to α might depend on additional considerations discussed by Shen (1966). For example, α could be chosen to minimize the discrepancy on a least-square basis between the gas variables obtained from the theory and the corresponding averages calculated from (15). Such refinements need further investigation. Here, for simplicity, we assume that α lies between $\frac{2}{3}$ and 1, from the well-known behaviour of the BGK model.

The generalized hydrodynamic equations thus arrived at are basically of the Navier–Stokes type since $(P_{ij})_R$ and $(\dot{q}_i)_R$ contain no more than the first derivatives of the macroscopic variables. Another essential feature of these equations is that they give a unified description of external rarefied flows at all Knudsen numbers and contain the two extreme Navier–Stokes and free-molecule limits. Thus they constitute the counterpart of the Navier–Stokes equations for external rarefied gas flows. This makes the treatment of a rarefied problem similar to, though more difficult than, the corresponding continuum one. In particular, continuum analytical solutions and computational procedures can be used as guidelines when dealing with our equations. These advantages will be illustrated in the following section for low-speed two-dimensional flows.

4. The governing equations for a low-speed two-dimensional flow

For an infinite cylinder of finite cross-section, we set up a cylindrical polar co-ordinate system centred on an axis parallel to the elements of the cylinder, the radial co-ordinate is being r and the polar angle θ. We restrict ourselves to a small molecular speed ratio, $S_\infty = U_\infty/(2RT_\infty)^{\frac{1}{2}} \ll 1$, and to a small temperature difference, $(T_w - T_\infty)/T_\infty \ll 1$. The subscripts ∞ and w denote the free-stream and wall conditions respectively. We then linearize the distribution function F_R with respect to the Maxwellian at infinity and consider only first-order terms in S_∞ and $(T_w - T_\infty)/T_\infty$. The collision frequency is taken to be $K = \alpha P_\infty/\mu_\infty$. Furthermore, we choose a length λ of the order of the mean free path as

$$\lambda = \alpha(2RT_\infty)^{\frac{1}{2}}/K. \tag{23}$$

This identifies α with K^* and leads after some rearrangement to

$$\lambda = (2S_\infty/Re)\, r_1, \tag{24}$$

where r_1 is some characteristic length of the directrix of the cylinder, and

$$Re = n_\infty U_\infty r_1/\mu_\infty$$

is the Reynolds number. We then define the Knudsen number as

$$Kn = \lambda/r_1 = 2S_\infty/Re. \tag{25}$$

Finally, we get the following linearized expression for F_R:

$$\begin{aligned} F_R = \frac{n_\infty}{(2\pi RT_\infty)^{\frac{3}{2}}}\, e^{-C^2} \Bigg\{ & (\tilde{n}_w - \tilde{n}^{(0)}) + (C^2 - \tfrac{3}{2})(\tilde{T}_w - \tilde{T}) - 2C_p S_\infty(u^* \cos\phi + v^* \sin\phi) \\ & + S_\infty\left(1 + \alpha\frac{S_p^0}{C_p}\right)\Bigg\{C_p^2\left[2\frac{\partial u^*}{\partial r^0}\cos^2\phi + 2\left(\frac{\partial v^*}{\partial r^0} + \frac{1}{r^0}\frac{\partial u^*}{\partial \theta} - \frac{v^*}{r^0}\right)\cos\phi\sin\phi \right. \\ & \left. + \frac{2}{r^0}\left(\frac{\partial v^*}{\partial \theta} + u^*\right)\sin^2\phi\right] \\ & - \frac{2}{5}\frac{\gamma}{\gamma - 1}\frac{1}{Pr\, S_\infty} C_p\,(\tfrac{5}{2} - C^2)\left(\frac{\partial \tilde{T}}{\partial r^0}\cos\phi + \frac{1}{r^0}\frac{\partial \tilde{T}}{\partial \theta}\sin\phi\right)\Bigg\}\Bigg\} + O(S_\infty^2), \end{aligned} \tag{26}$$

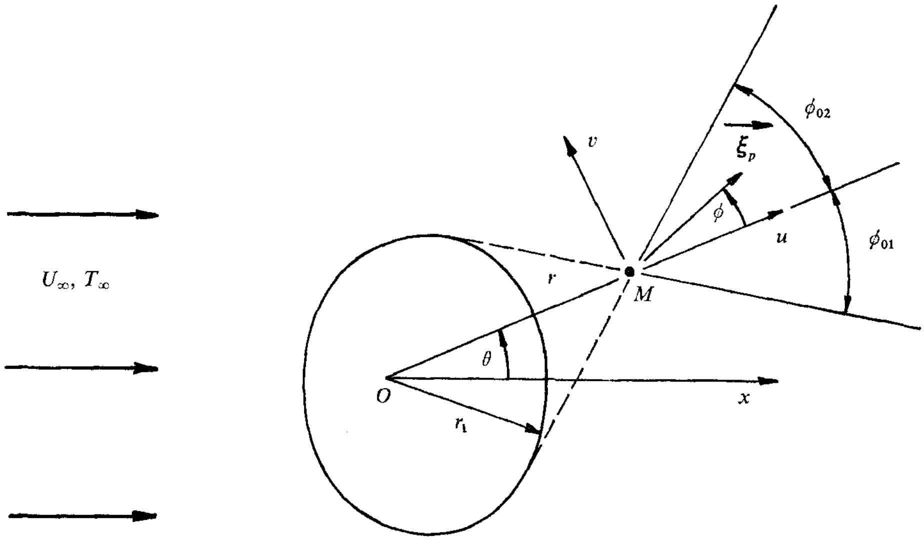

FIGURE 3. Geometrical configuration in the cross-sectional plane of the cylinder. $\boldsymbol{\xi}_p$ is the projection of the molecular velocity $\boldsymbol{\xi}$ and ϕ is the polar angle in the phase space. The angles ϕ_{01} and ϕ_{02} define the planes that delimit the semi-space A.

where $u^* = u/U_\infty$ and $v^* = v/U_\infty$ are the normalized gas velocity components in the cylindrical co-ordinate system, $C = \xi/(2RT_\infty)^{\frac{1}{2}}$ is the normalized molecular velocity, ϕ stands for the polar angle in the phase space shown on figure 3, C_p and S_p^0 are the projections of C and S^0 on the r, θ plane and $r^0 = r/\lambda$. The quantities $\tilde{n}_w$, $\tilde{n}^{(0)}$, $\tilde{T}$ and $\tilde{T}_w$ are the perturbation values defined by

$$\tilde{n}_w = \frac{n_w - n_\infty}{n_\infty}, \quad \tilde{n}^{(0)} = \frac{n^{(0)} - n_\infty}{n_\infty}, \quad \tilde{T} = \frac{T - T_\infty}{T_\infty}, \quad \tilde{T}_w = \frac{T_w - T_\infty}{T_\infty}. \tag{27}$$

Finally, Pr and γ represent the Prandtl number and the specific heat ratio respectively.

The complementary elements of the stress tensor $(P_{ij})_R$ and heat flux vector $(\dot{q}_i)_R$ are then calculated from (19) after linearization with respect to the flow velocity u_i and substitution of F_R by its linearized expression (26). By introducing the dimensionless quantities

$$\tilde{P}_{ij} = -P_{ij}/2n_\infty RT_\infty, \quad \tilde{q}_i = \dot{q}_i/n_\infty RT_\infty(2RT_\infty)^{\frac{1}{2}}, \tag{28}$$

we get after some rearrangement

$$\begin{aligned}\tilde{P}_{rr} = {} & G_{320}(\tilde{n}_w - \tilde{n}^{(0)}) + (G_{520} - G_{320})(\tilde{T}_w - \tilde{T}) - 2S_\infty(G_{430}u^* + G_{421}v^*) \\ & + 2S_\infty\left[L_{540}\frac{\partial u^*}{\partial r^0} + \left(\frac{\partial v^*}{\partial r^0} + \frac{1}{r^0}\frac{\partial u^*}{\partial \theta} - \frac{v^*}{r^0}\right)L_{531} + L_{522}\left(\frac{u^*}{r^0} + \frac{1}{r^0}\frac{\partial v^*}{\partial \theta}\right)\right] \\ & - \frac{2}{5}\frac{\gamma}{\gamma - 1}\frac{1}{Pr}\left[(2L_{430} - L_{630})\frac{\partial \tilde{T}}{\partial r^0} + (2L_{421} - L_{621})\frac{1}{r^0}\frac{\partial \tilde{T}}{\partial \theta}\right],\end{aligned} \tag{29}$$

$$\begin{aligned}\tilde{P}_{\theta\theta} = {} & G_{302}(\tilde{n}_w - \tilde{n}^{(0)}) + (G_{502} - G_{302})(\tilde{T}_w - \tilde{T}) - 2S_\infty(G_{412}u^* + G_{403}v^*) \\ & + 2S_\infty\left[L_{522}\frac{\partial u^*}{\partial r^0} + \left(\frac{\partial v^*}{\partial r^0} + \frac{1}{r^0}\frac{\partial u^*}{\partial \theta} - \frac{v^*}{r^0}\right)L_{513} + L_{504}\left(\frac{u^*}{r^0} + \frac{1}{r^0}\frac{\partial v^*}{\partial \theta}\right)\right] \\ & - \frac{2}{5}\frac{\gamma}{\gamma - 1}\frac{1}{Pr}\left[(2L_{412} - L_{612})\frac{\partial \tilde{T}}{\partial r^0} + (2L_{403} - L_{603})\frac{1}{r^0}\frac{\partial \tilde{T}}{\partial \theta}\right],\end{aligned} \tag{30}$$

$$\tilde{P}_{r\theta} = G_{311}(\tilde{n}_w - \tilde{n}^{(0)}) + (G_{511} - G_{311})(\tilde{T}_w - \tilde{T}) - 2S_\infty(G_{421}u^* + G_{412}v^*)$$
$$+ 2S_\infty\left[L_{531}\frac{\partial u^*}{\partial r^0} + \left(\frac{\partial v^*}{\partial r^0} + \frac{1}{r^0}\frac{\partial u^*}{\partial \theta} - \frac{v^*}{r^0}\right)L_{522} + L_{513}\left(\frac{u^*}{r^0} + \frac{1}{r^0}\frac{\partial v^*}{\partial \theta}\right)\right]$$
$$-\frac{2}{5}\frac{\gamma}{\gamma-1}\frac{1}{Pr}\left[(2L_{421} - L_{621})\frac{\partial \tilde{T}}{\partial r^0} + (2L_{412} - L_{612})\frac{1}{r^0}\frac{\partial \tilde{T}}{\partial \theta}\right], \quad (31)$$

$$(\tilde{q}_r)_R = (G_{410} + \tfrac{1}{2}G_{210})(\tilde{n}_w - \tilde{n}^{(0)}) + (G_{610} - \tfrac{1}{2}G_{410})(\tilde{T}_w - \tilde{T}) - 2S_\infty[(G_{520} + \tfrac{1}{2}G_{320})u^*$$
$$+ (G_{511} + \tfrac{1}{2}G_{311})v^*] + 2S_\infty\left[(L_{630} + \tfrac{1}{2}L_{430})\frac{\partial u^*}{\partial r^0}\right.$$
$$\left. + (L_{621} + \tfrac{1}{2}L_{421})\left(\frac{\partial v^*}{\partial r^0} + \frac{1}{r^0}\frac{\partial u^*}{\partial \theta} - \frac{v^*}{r^0}\right) + (L_{612} + \tfrac{1}{2}L_{412})\left(\frac{1}{r^0}\frac{\partial v^*}{\partial \theta} + \frac{u^*}{r^0}\right)\right]$$
$$-\frac{2}{5}\frac{\gamma}{\gamma-1}\frac{1}{Pr}\left[(\tfrac{3}{2}L_{520} + \tfrac{1}{2}L_{320} - L_{720})\frac{\partial \tilde{T}}{\partial r^0} + (\tfrac{3}{2}L_{511} + \tfrac{1}{2}L_{311} - L_{711})\frac{\partial \tilde{T}}{r^0\partial \theta}\right], \quad (32)$$

$$(\tilde{q}_\theta)_R = (G_{401} + \tfrac{1}{2}G_{201})(\tilde{n}_w - \tilde{n}^{(0)}) + (G_{601} - \tfrac{1}{2}G_{401})(\tilde{T}_w - \tilde{T}) - 2S_\infty[(G_{511} + \tfrac{1}{2}G_{311})u^*$$
$$+ (G_{502} + \tfrac{1}{2}G_{302})v^*] + 2S_\infty\left[(L_{621} + \tfrac{1}{2}L_{421})\frac{\partial u^*}{\partial r^0}\right.$$
$$\left. + (L_{612} + \tfrac{1}{2}L_{412})\left(\frac{\partial v^*}{\partial r^0} + \frac{1}{r^0}\frac{\partial u^*}{\partial \theta} - \frac{v^*}{r^0}\right) + (L_{603} + \tfrac{1}{2}L_{403})\left(\frac{1}{r^0}\frac{\partial v^*}{\partial \theta} + \frac{u^*}{r^0}\right)\right]$$
$$-\frac{2}{5}\frac{\gamma}{\gamma-1}\frac{1}{Pr}\left[(\tfrac{3}{2}L_{511} + \tfrac{1}{2}L_{311} - L_{711})\frac{\partial \tilde{T}}{\partial r^0} + (\tfrac{3}{2}L_{502} + \tfrac{1}{2}L_{302} - L_{702})\frac{1}{r^0}\frac{\partial \tilde{T}}{\partial \theta}\right]. \quad (33)$$

Similarly the additional equation for the density is calculated from (20):

$$n = n^{(0)} + n_\infty\left\{G_{100}(\tilde{n}_w - \tilde{n}^{(0)}) + (G_{300} - G_{100})(\tilde{T}_w - \tilde{T}) - 2S_\infty(G_{210}u^* + G_{201}v^*)\right.$$
$$+ 2S_\infty\left[L_{320}\frac{\partial u^*}{\partial r^0} + \left(\frac{\partial v^*}{\partial r^0} + \frac{1}{r^0}\frac{\partial u^*}{\partial \theta} - \frac{v^*}{r^0}\right)L_{311} + L_{302}\left(\frac{u^*}{r^0} + \frac{1}{r^0}\frac{\partial v^*}{\partial \theta}\right)\right]$$
$$\left.-\frac{2}{5}\frac{\gamma}{\gamma-1}\frac{1}{Pr}\left[(2L_{210} - L_{410})\frac{\partial \tilde{T}}{\partial r^0} + (2L_{201} - L_{401})\frac{\partial \tilde{T}}{r^0\partial \theta}\right]\right\}, \quad (34)$$

where
$$G_{ijk} = \frac{1}{\pi}\int_{\phi_{01}}^{\phi_{02}} \cos^j\phi \sin^k\phi J_i(S_p^0)\,d\phi, \quad (35)$$

$$L_{ijk} = G_{ijk} + \frac{1}{\pi}\int_{\phi_{01}}^{\phi_{02}} S_p^0 \cos^j\phi \sin^k\phi\, J_{i-1}(S_p^0)\,d\phi, \quad (36)$$

$$J_i(S_p^0) = \int_0^\infty x^i \exp -\left(x^2 + \frac{S_p^0}{x}\right)dx. \quad (37)$$

The angles ϕ_{01} and ϕ_{02} are those delimiting the semi-space A as shown in figure 3. The functions G_{ijk} and L_{ijk} depend, in general, on the two space variables r and θ.

The boundary conditions at the wall likewise follow from (21):

$$(\tilde{n}-\tilde{n}^{(0)})\,S_\infty u^* = \frac{1}{4\sqrt{\pi}}S_{10}(\tilde{n}_w-\tilde{n}^{(0)})+\frac{1}{8\sqrt{\pi}}S_{10}(\tilde{T}_w-\tilde{T})-S_\infty\frac{1}{\pi}(S_{20}u^*+S_{11}v^*)$$
$$+\frac{3}{4\sqrt{\pi}}S_\infty\left[S_{30}\frac{\partial u^*}{\partial r^0}+\left(\frac{\partial v^*}{\partial r^0}+\frac{1}{r^0}\frac{\partial u^*}{\partial \theta}-\frac{v^*}{r^0}\right)S_{21}+S_{12}\left(\frac{u^*}{r^0}+\frac{1}{r^0}\frac{\partial v^*}{\partial \theta}\right)\right], \tag{38}$$

$$(\tilde{n}-\tilde{n}^{(0)})\,S_\infty v^* = \frac{1}{4\sqrt{\pi}}S_{01}(\tilde{n}_w-\tilde{n}^{(0)})+\frac{1}{8\sqrt{\pi}}S_{01}(\tilde{T}_w-\tilde{T})-\frac{S_\infty}{\pi}(S_{11}u^*+S_{02}v^*)$$
$$+\frac{3}{4\sqrt{\pi}}S_\infty\left[S_{21}\frac{\partial u^*}{\partial r^0}+S_{12}\left(\frac{\partial v^*}{\partial r^0}+\frac{1}{r^0}\frac{\partial u^*}{\partial \theta}-\frac{v^*}{r^0}\right)+S_{03}\left(\frac{u^*}{r^0}+\frac{1}{r^0}\frac{\partial v^*}{\partial \theta}\right)\right], \tag{39}$$

$$(\tilde{n}-\tilde{n}^{(0)})\frac{T}{T_\infty} = \frac{2}{3}\left\{\frac{3}{4\pi}S_{00}(\tilde{n}_w-\tilde{n}^{(0)})+\frac{3}{4\pi}S_{00}(\tilde{T}_w-\tilde{T})-2S_\infty\left[\frac{1}{2\sqrt{\pi}}(S_{10}u^*+S_{01}v^*)\right]\right.$$
$$+\frac{5}{2\pi}S_\infty\left[S_{20}\frac{\partial u^*}{\partial r^0}+S_{11}\left(\frac{\partial v^*}{\partial r^0}+\frac{1}{r^0}\frac{\partial u^*}{\partial \theta}-\frac{v^*}{r^0}\right)+S_{02}\left(\frac{1}{r^0}\frac{\partial v^*}{\partial \theta}+\frac{u^*}{r^0}\right)\right]$$
$$\left.+\frac{2}{5}\frac{\gamma}{\gamma-1}\frac{1}{Pr}\frac{1}{4\sqrt{\pi}}\left(S_{10}\frac{\partial \tilde{T}}{\partial r^0}+S_{01}\frac{\partial \tilde{T}}{r^0\partial\theta}\right)\right\}, \tag{40}$$

where all the quantities take the values for the gas at the wall, and

$$S_{jk} = \int_{\phi_{w1}}^{\phi_{w2}} \cos^j\phi \sin^k\phi\, d\phi, \tag{41}$$

where ϕ_{w1} and ϕ_{w2} are the limits of ϕ_{01} and ϕ_{02} respectively as the space point M tends to the surface of the cylinder. These angles are easily obtained from the geometry of the body. Let $r = g(\theta)$ be the equation of the directrix of the cylinder, then $\phi_{w2} = \text{arc tan}\,(g(\theta)/g'(\theta))$ and $\phi_{w1} = \phi_{w2}-\pi$. Finally, let (ν_1, ν_2) and (τ_1, τ_2) be the components in the polar co-ordinate system (r, θ) of the unit vectors normal and tangential to the directrix of the cylinder respectively. Then the conservation of mass at an impermeable wall yields

$$\nu_1 u^* + \nu_2 v^* = 0. \tag{42}$$

This equation will serve to determine the density at the wall $\tilde{n}_w$, while (40) and the linear combination $\tau_1 u^* + \tau_2 v^*$ of (38) and (39) will provide the jump and slip conditions at the wall, respectively.

As in the continuum regime, the convective terms in (17) may also be linearized with respect to S_∞. The linearized system thus obtained will provide us with the solution for the limiting case of vanishing Mach number. In connexion with this there is the interesting question of when and in what sense a rarefied gas may be treated as incompressible. It seems, then, reasonable to follow the same procedure as in the continuum regime and to rewrite the full system of equations (17) in a non-dimensional form and order the terms in S_∞. However, we have an additional difficulty in finding the proper length unit since the rarefied terms depend at the same time on two fundamental lengths: the characteristic geometric length r_1 and the mean free path λ. For a given point at a distance r from the cylinder, $(P_{ij})_R$ and $(\dot{q}_i)_R$ depend on the solid angle delimiting the semi-space A, and thus on r_1/r. On the other hand, the exponential attenuation factor E

depends on the distance travelled by the molecules in terms of mean free paths, S/λ. So we shall discuss three cases according to the order of the Knudsen number.

(i) If $Kn = O(1)$, this also means that $S_\infty = O(Re)$. The proper length unit is the mean free path. If we further choose

$$P_{ij}^0 = \frac{P_{ij}}{(\mu U_\infty/\lambda)}, \quad q_i^0 = \frac{\dot{q}_i}{(kT_\infty/\lambda)} \tag{43}$$

as new variables and note that $P_\infty/(\mu U_\infty/\lambda) = 1/S_\infty$, then equations (17) become

$$\left.\begin{aligned} \frac{D}{Dt^0}\tilde{n} + (1+\tilde{n})\frac{\partial u_i^*}{\partial x_i^0} &= 0, \\ 2S_\infty(1+\tilde{n})\frac{D}{Dt^0}u_j^* - \frac{\partial}{\partial x_i}[(P_{ij}^0)_C + (P_{ij}^0)_R] &= 0, \\ \frac{S_\infty}{\gamma-1}\frac{D}{Dt^0}\tilde{T} - S_\infty^2[(P_{ij}^0)_C + (P_{ij}^0)_R]\frac{\partial u_i^*}{\partial x_j^0} + \frac{\gamma}{\gamma-1}\frac{1}{2Pr}\frac{\partial}{\partial x_i^0}[(q_i^0)_C + (q_i^0)_R] &= 0, \end{aligned}\right\} \tag{44}$$

where $$\frac{D}{Dt^0} \equiv \frac{\partial}{\partial t^0} + u^*\frac{\partial}{\partial x_i^0}, \quad t^0 = \frac{U_\infty}{\lambda}t, \quad x_i^0 = \frac{x_i}{\lambda}.$$

When $S_\infty \to 0$, the leading terms in the energy equation are the conduction terms

$$\frac{\partial}{\partial x_i^0}[(q_i^0)_C + (q_i^0)_R] = O(S_\infty^2). \tag{45}$$

The choice of the new variables defined by (43) will not, in general, be possible because of the density and temperature terms in the expressions of $(P_{ij}^0)_R$ and $(q_i^0)_R$, unless these terms are also of order S_∞. This is determined by the boundary conditions at the wall. When we have an isothermal wall ($T_w = T_\infty$) or an adiabatic wall ($\dot{q}_w = 0$), then it can be seen from (45) and the expressions for the heat flux vector and the boundary conditions that

$$\tilde{T} = O(S_\infty), \quad \tilde{n}^{(0)} = O(S_\infty). \tag{46}$$

This shows that for a rarefied gas flow over an isothermal or adiabatic body compressibility effects are of the order of the Mach number, whereas they are of $O(M_\infty^2)$ in the continuum case. This remarkable result for rarefied gas flows comes from the departure of the local velocity distribution function from the Maxwellian equilibrium. Its effect is maximum at the wall and is responsible for the temperature jump and velocity slip.

(ii) For $Kn \ll 1$, the rarefaction effects are limited to a thin layer called the Knudsen layer. When $T_w = T_\infty$ or $\dot{q}_w = 0$, outside the Knudsen layer we have $\nabla^2\tilde{T} = O(S_\infty^2)$. Within the Knudsen layer, we can get a fast estimate of the order of $\tilde{T}$ if, at a given point of the surface of the cylinder, the co-ordinates (r, θ) are centred at its centre of curvature. The basic form of (29)–(40) would be the same, with u^* and v^* being the normal and tangential velocities. Then, from (38) we see that the jump temperature $[\tilde{T}]_J = O(S_\infty Kn\, v^*)$. Since the slip velocity v^* is $O(Kn)$ we conclude that $\tilde{T} = O(S_\infty Kn^2)$. The order of $\tilde{T}$ will then be the larger of

S_∞^2 and $S_\infty Kn^2$. On the other hand, the variations of the pressure and density are of order

$$\frac{\mu U_\infty / r_1}{n_\infty R T_\infty} = Kn\, S_\infty.$$

(iii) For $Kn \gg 1$, the body appears as a singular point in (44). These equations should only describe the far field. Near the body, we have rather a slightly perturbed free-molecule flow and the compressibility effects are of $O(S_\infty)$ for a nearly isothermal or adiabatic body.

It should be pointed out that, though the above discussion, for simplicity, was based on the linearized expressions for $(P_{ij})_R$ and $(\dot{q}_i)_R$ for two-dimensional flows, the conclusions are equally valid for three-dimensional flows.

In view of the above results, for a nearly isothermal or adiabatic wall and as a first approximation to $O(S_\infty)$, equations (44) are reduced to

$$\left.\begin{aligned} \partial u_i^* / \partial x_i^0 = 0,\\ \frac{\partial}{\partial x_i^0}(P_{ij}^0)_C + \frac{\partial}{\partial x_i^0}(P_{ij}^0)_R = 0. \end{aligned}\right\} \tag{47}$$

These equations are the counterpart of the Stokes equations in the continuum regime. However, the approximation which led to (47) is not uniformly valid. When $Kn \geqslant O(1)$, which also means that $Re \ll 1$, the non-uniformity arises in the far field. As in the continuum regime, this is because at large distance from the body the proper length unit is the viscous length ν/U_∞ and no longer the geometric length r_1 or the mean free path λ. In fact, if ν/U_∞ is taken as a length unit and we note that $\lambda/(\nu/U_\infty) = 2S_\infty$, then it becomes obvious that the convective terms in the momentum equations can no longer be neglected. For $Kn \ll 1$, equations (47) describe the flow only within the thin Knudsen layer. Outside this layer, they tend to the Stokes equations. The only difference then from the continuum regime is that the inner boundary conditions are modified to $O(Kn)$ by the slip velocity. Consequently, the results of the continuum regime analysis (see Lagerstrom 1964, Van Dyke 1964) apply entirely to the case of a small Knudsen number. We then conclude that outside the Knudsen layer our linearization procedure is only valid for small Reynolds numbers. Besides, for two-dimensional flows, the solutions of the Stokes equations are non-uniform at infinity. We have then to resort to a matching procedure with outer solutions obtained from the Oseen equation.

This non-uniformity at large distance from the body is equally present in the solutions of the linearized Boltzmann equation. Cercignani (1969) discussed this problem and concluded that, in particular for two-dimensional flows, there is no uniform solution to the distribution function. However, the incompressible form of the continuity equation is uniformly valid in the plane and we can introduce a stream function ψ defined by

$$u = (1/r)\,\psi_\theta, \quad v = -\psi_r. \tag{48}$$

Only one governing equation is obtained by introducing (48) into (44), and after returning to the physical variables we have

$$\left[\nabla^2 - \frac{n_\infty}{\mu}\left(\frac{\partial}{\partial t} + \frac{1}{r}\psi_\theta \frac{\partial}{\partial r} - \frac{1}{r}\psi_r \frac{\partial}{\partial \theta}\right)\right]\nabla^2\psi = \frac{\mathscr{R}}{\mu r}. \tag{49}$$

where $$\mathscr{R} = \frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(P_{r\theta})_R + \frac{\partial}{\partial \theta}(P_{\theta\theta})_R + 2(P_{r\theta})_R\right] - \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}(P_{rr})_R + \frac{\partial}{r\,\partial \theta}(P_{r\theta})_R + \frac{(P_{rr})_R - (P_{\theta\theta})_R}{r}\right]. \quad (50)$$

Solutions of (49) will depend on the Knudsen number. In part 2 of this paper we shall develop analytical solutions to (49) for near-continuum and nearly free-molecule flows past a circular cylinder. For Knudsen numbers of order unity a finite-difference numerical solution is being developed by Atassi. However, the great advantage of our formulation is that all flow regimes are described by equation (49), which is basically similar to the classical Navier–Stokes equation and can be solved within the same numerical framework.

5. Concluding remarks

The theory we have formulated for external rarefied gas flows leads to a system of partial differential equations of the Navier–Stokes type. Our approach is not rigorously derived from the Boltzmann equation. Accordingly we do not expect to reproduce the large amount of details of the two-body interaction contained in the collision integral of that equation. However, to aerodynamicists, the main interest in solving an external flow problem is to determine such quantities as the drag and heat-transfer rate and to obtain a sufficiently accurate description of the flow pattern over the body for further use in design. These are not likely to be significantly influenced by such fine details.

Our concern is now to show through applications the accuracy of our theory and the relative simplification brought about by our formulation. The effect of neglecting the term $\tilde{F}$ in (9) is to make the distribution function F an interpolation between the nearly free-molecule and near-continuum regimes which retains only the qualitative and average properties of the true distribution functions. However, because of the contact conditions imposed on $\tilde{F}$ at both limits, we expect our theory to reproduce, at least to a first order, these two asymptotic regimes. This will be shown in part 2 of this paper to be true for a low-speed flow over a circular cylinder. On the other hand, our equations of the Navier–Stokes type have the interesting property of making the rarefied problem the counterpart of the continuum one. Thus we can adapt and use the various analytical and numerical methods and techniques developed for handling the Navier–Stokes equations to solve external rarefied problems. From this viewpoint, our analytical treatment of a low-speed rarefied flow past a circular cylinder will appear as an extension to the classical work of Proudman & Pearson (1957). In the transition regime, where only numerical solutions are possible, our equations can be solved within the same numerical framework developed for the Navier–Stokes equations; the introduction of the complementary rarefied terms is not expected to change, basically, the computational procedure.

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